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An asymptotically correct model for initially curved and twisted thin-walled composite beams ☆

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Abstract

The variational–asymptotic method has been applied to develop an asymptotically correct model for initially curved and twisted, thin-walled, composite beams of arbitrary cross-sectional shapes and arbitrary anisotropic materials. In a two-step asymptotic reduction procedure, the three-dimensional strain energy is asymptotically reduced first to a two-dimensional shell strain energy and then to a one-dimensional beam strain energy. This is a new attempt where initially curved and twisted, thin-walled, composite beams, with open or closed sections, have been modeled in an asymptotically correct unified framework.

Keywords: Initial curvature/twist; Thin-walled; Composite beams; Variational–asymptotic method

1. Introduction

In spite of the fact that tremendous advancement in the computational power has made it possible for three-dimensional (3D) numerical analysis of structures with complex geometry and general anisotropic materials, technical community still today relies on simplified structural models for efficiently getting applicable results and for understanding the specific behavioral pattern resulting from the geometrical features of the structure. Such simplified models are particularly helpful in the stages of preliminary design. However, simplicity and efficiency are not the only goals of such models. To carry out meaningful design tradeoffs, such models must be accurate, or predictive, for the behavior of real structures. Here, we are presenting a method to construct *an accurate yet efficient model* for initially curved and twisted, thin-walled, composite beams having arbitrary cross-sectional shapes.

Thin-walled beams are classified as the flexible structures having two geometric features: the characteristic dimension of the cross-section (c) is much smaller than the wave length of the deformation along the axis (l),

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i.e., $\frac{c}{l} \ll 1$ (beam definition) and the characteristic wall thickness (h) is much smaller than the cross-sectional dimension (c), i.e., $\frac{h}{c} \ll 1$ (thin-walled definition). Thin-walled beam theories take advantage of the small parameters, h/c and c/l, to derive a one-dimensional (1D) model, which provides 1D constitutive models for 1D beam analysis and recovery relations to back calculate the 3D fields within the structures based on the global behavior predicted by the 1D beam analysis.

Numerous models have been developed for thin-walled composite beams over the last several decades. A review and analysis of various theories can be obtained in Volovoi et al. (2001). Generally speaking, most of thin-walled beam models fall into one of two groups: viz., *ad hoc* models (Vlasov, 1961; Gjelsvik, 1981; Chandra and Chopra, 1991) and asymptotic models (Badir et al., 1993; Volovoi et al., 1999; Volovoi and Hodges, 2000). *Ad hoc* models assume a priori certain forms for the unknown displacement field or stress field in terms of beam variables and cross-sectional coordinates. Asymptotic models start from the original 3D formulation for anisotropic elasticity and rigorously reduce the dimensions of the problem into 1D formulations using small parameters inherent to the structure. While application of traditional asymptotic methods is possible, we will adopt the variational–asymptotic method (VAM) (Berdichevsky, 1979), which combines both merits of variational method and asymptotic method, to carry out the asymptotic expansion in a systematic manner.

This paper is organized in the following manner. First, we present the kinematical fundamentals to exactly describe the geometry of initially curved and twisted thin-walled beams. Then, the 3D strain energy of the structure is reduced to the two-dimensional (2D) strain energy corresponding to the classical shell approximation in Berdichevsky (1979) with geometric correction by considering h/c as the main small parameter and taking into account all first-order corrections from the initial twist and curvatures of the thin-walled beam. By expressing the 2D shell variables in terms of intrinsic beam variables and unknown warping functions, we can use VAM to solve for the unknown warping functions and finally reduce the 3D formulation into a 1D beam formulation. The final result is a 1D strain energy in the form of Euler–Bernoulli model for the thin-walled composite beams with first-order corrections from initial twist and curvatures. Several examples are used to demonstrate application and accuracy of the present theory.

The present work is built upon Yu et al. (2005), and makes unique contributions to the subject in the following three aspects:

- For an initially curved and twisted beam, the reference line becomes a general 3D space curve, while for initially twisted beams, the reference line is a straight line. Rigorous kinematical descriptions of initially curved and twisted structures make the present work much more challenging than that where only initially twisted structures were considered by Yu et al. (2005).
- It is shown that the first-order correction to the strain energy comes from the 3D material tensor *and* 3D mechanical strains. The first-order correction in the material tensor comes mainly due to the presence of initial curvatures. A systematic transformation rule has been developed in the present work to transform the general anisotropic material tensor defined in the material coordinates into the global coordinates. While, such a transformation will not significantly affect the results of Yu et al. (2005).
- Regarding practical applications, the present work can analyze both initially curved and/or twisted composite structures which are beyond the capability of Yu et al. (2005).

Comparing to the existing initially twisted (twisted turbine blades, windmills, propellers, and helicopter blades) *or* curved structures (arches and curved bridges), initially curved *and* twisted structures are not so common. Nevertheless, there are some examples such as the twisted and curved slides in playground, ramps of many-storied garage, and helical stairs, which can be designed and analyzed using the model of initially curved and twisted structures. It is possible that the existing initially curved *or* twisted structures may one day be designed as initially curved *and* twisted if the engineers are equipped with more efficient yet predictive models as the one developed here.

2. Kinematical fundamentals

The geometry of a general thin-walled beam structure is drawn in Fig. 1. Please note that for simplicity and clarity, initial curvatures, and twist are not sketched in figure. The geometry is prescribed by single parameter

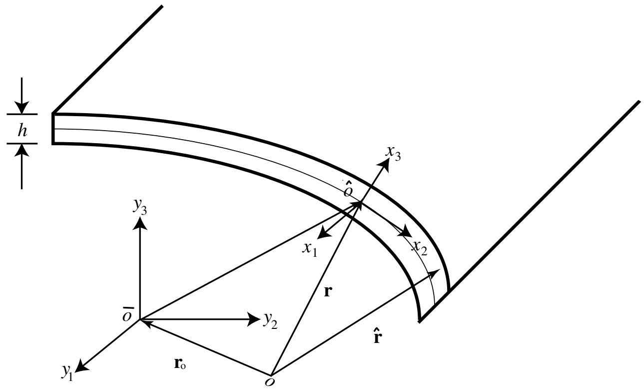

Fig. 1. Schematic of a thin-walled beam.

(x_1 or y_1) vector equation of a space curve (beam reference line) and double parameter (x_α) vector equation of a surface (shell reference surface). In the figure, O is an inertial point fixed in space, $\overline{O}$ is shifting its position along the beam axis located by the position vector $\mathbf{r}_0$, and $\hat{O}$ is shifting its position along the contour intersecting the reference surface (considering the thin-walled structure as a shell) with the beam section cut through the point $\overline{O}$. Here, two right-hand coordinate systems x_i and y_i are defined: y_1 is the arc length measured along the beam axis with $\mathbf{b}_1$ as the unit vector; y_a are the local cross-sectional Cartesian coordinates (locating $\hat{O}$ with respect to $\overline{O}$) of the beam section with $\mathbf{b}_a$ as the unit vectors; x_1 is parallel to y_1, x_3 is the outward normal of the reference surface and x_2 is the arc length along the contour; this is the *definition* of *parametric space* we are using. Here and throughout this paper, Greek indices take values 1 and 2 while Latin indices $i,j,\ldots,z$ assume 1, 2, and 3, and $a,b,\ldots,h$ assume 2 and 3. Repeated indices follow the summation convention unless otherwise mentioned. It is noted that unit vectors $\mathbf{b}_i$ are functions of y_1.

The position vector of an arbitrary material point on the shell reference surface with respect to the inertial reference point O is given by the vector equation

$$\mathbf{r}(x_1,x_2) = \mathbf{r}_0(y_1) + y_a(x_2)\mathbf{b}_a \tag{1}$$

The covariant base vectors of the undeformed reference surface can be obtained using

$$\mathbf{a}_\alpha = \mathbf{r}_{,\alpha} \tag{2}$$

where $(\bullet)_{,\alpha} = \frac{\partial(\bullet)}{\partial x_\alpha}$. Eq. (1) gives the explicit form of the covariant base vectors as

$$\begin{aligned}
\mathbf{a}_1 &= (1 - k_3 y_2 + k_2 y_3)\mathbf{b}_1 + y_2 k_1 \mathbf{b}_3 - y_3 k_1 \mathbf{b}_2 \\
\mathbf{a}_2 &= \dot{y}_a \mathbf{b}_a \\
\mathbf{a}_3 &= \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}
\end{aligned} \tag{3}$$

where $(\dot{\bullet}) = \frac{\partial(\bullet)}{\partial x_2}$, k_1 is the initial twist, and k_a are the initial curvatures. In deriving Eq. (3), we made use of the following relations of the moving triad $\mathbf{b}_i$ with respect to k_i, which is called *Frenet's* description of space curve in differential geometry, given as

$$\begin{aligned}
\mathbf{b}_1' &= k_3 \mathbf{b}_2 - k_2 \mathbf{b}_3 \\
\mathbf{b}_2' &= -k_3 \mathbf{b}_1 + k_1 \mathbf{b}_3 \\
\mathbf{b}_3' &= k_2 \mathbf{b}_1 - k_1 \mathbf{b}_2
\end{aligned} \tag{4}$$

where $(\bullet') = \frac{\partial(\bullet)}{\partial y_1}$. The first fundamental form of the surface is given by $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ as follows:

$$\begin{aligned} a_{11} &= (1 - k_3 y_2 + k_2 y_3)^2 + k_1^2 R^2 \\ a_{12} &= a_{21} = k_1 R_n \\ a_{22} &= 1 \end{aligned} \tag{5}$$

with $R^2 = y_a \dot{y}_a$ and $R_n = y_2 \dot{y}_3 - y_3 \dot{y}_2$. The second fundamental form of the undeformed reference surface is calculated by the formula

$$b_{\alpha\beta} = \mathbf{r}_{,\alpha\beta} \cdot \mathbf{a}_3 = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}_3 \tag{6}$$

It is important to note that the geometry of any smooth surface (i.e., any function defined on it is infinitely differentiable) is completely governed by these two fundamental forms except rigid body motions. For convenience of compact expression, other forms of fundamental forms (changing the position of the free indices) are conveniently used. Such as $b^{\beta}_{\cdot\alpha} = a^{\beta\nu} b_{\nu\alpha}$ and $b^{\beta\alpha} = a^{\beta\gamma} a^{\alpha\nu} b_{\gamma\nu}$, with $a^{\beta\alpha} = (a_{\beta\alpha})^{-1}$.

An arbitrary material point in the 3D structure is located with respect to the inertial point O by the position vector

$$\hat{\mathbf{r}}(x_1, x_2, x_3) = \mathbf{r}(x_1, x_2) + x_3 \mathbf{a}_3 \tag{7}$$

The 3D covariant base vectors are defined using $\mathbf{g}_i = \hat{\mathbf{r}}_{,i}$ and given as

$$\begin{aligned} \mathbf{g}_\alpha &= \mathbf{a}_\alpha - x_3 b^{\cdot\lambda}_{\alpha} \mathbf{a}_\lambda \\ \mathbf{g}_3 &= \mathbf{a}_3 \end{aligned} \tag{8}$$

The 3D metric tensor is given by

$$\begin{aligned} g_{\alpha\beta} &= a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 b^{\cdot\lambda}_{\alpha} b_{\lambda\beta} \\ g_{i3} &= \delta_{i3} \end{aligned} \tag{9}$$

and $\mathrm{Det}(g_{ij}) = [1 - 2x_3 H + x_3^2 K]^2 \mathrm{Det}(a_{\alpha\beta})$ with $H = \frac{1}{2} b^{\alpha}_{\alpha}$ as the mean curvature and $K = \mathrm{Det}(b^{\beta}_{\alpha})$ as the Gaussian curvature of the surface.

3. Dimensional reduction from 3D to 2D

The dimensional reduction from the original 3D formulation of the thin-walled structure to a 1D beam model can be carried out in two steps due to the existence of two different small parameters h/c and c/l. First, making use of h/c, one can asymptotically approximate the original 3D energy to a 2D energy defined in the shell reference surface. Second, making use of c/l, one can approximate the above-found 2D energy in an asymptotical sense to a 1D energy defined along the beam axis. To reduce the dimension from 3D to 2D, we need to formulate the 3D problem in terms of 2D variables, which is shown in this section.

3.1. Deformed kinematics

A generic material point in the deformed configuration can be located by the position vector

$$\hat{\mathbf{R}}(x_1, x_2, x_3) = \mathbf{R}(x_1, x_2) + x_3 \mathbf{A}_3(x_1, x_2) + w_i(x_1, x_2, x_3) \mathbf{A}^i(x_1, x_2) \tag{10}$$

In Eq. (10), the $\mathbf{A}^i$ are the contravariant counterparts of $\mathbf{A}_i$, the base vectors of the deformed reference surface, which are defined as

$$\begin{aligned} \mathbf{A}_\alpha &= \mathbf{R}_{,\alpha} \\ \mathbf{A}_3 &= \frac{\mathbf{A}_1 \times \mathbf{A}_2}{|\mathbf{A}_1 \times \mathbf{A}_2|} \end{aligned} \tag{11}$$

Here, we choose $\mathbf{A}_3$ to be the normal to the deformed reference surface by including all possible distortion of the transverse normal into the 3D warping functions w_i. One can define $\mathbf{R}$ as

$$\mathbf{R} = \frac{1}{h}\langle \hat{\mathbf{R}} \rangle \tag{12}$$

The definition in Eq. (12) makes the vector functional transformation in Eq. (10) unique and boils down to three mechanical constrains on the unknown 3D warping functions, given as

$$\langle w_i(x_1, x_2, x_3) \rangle = 0 \tag{13}$$

where the angle bracket denotes the definite integral through the thickness of the shell. The 3D covariant base vectors and the metric are defined by

$$\begin{aligned} \mathbf{G}_i &= \hat{\mathbf{R}}_{,i} \\ G_{ij} &= \mathbf{G}_i \cdot \mathbf{G}_j \end{aligned} \tag{14}$$

The 3D Lagrangian strain tensor is defined as

$$\Gamma_{ij} = \tfrac{1}{2}(G_{ij} - g_{ij}) \tag{15}$$

For the purpose of constructing a linear material model, the 3D strains could be explicitly written with terms linear with respect to $\epsilon_{\alpha\beta}$, $\kappa_{\alpha\beta}$, and w_i.

$$\begin{aligned} \Gamma_{\alpha\beta} &= \epsilon_{\alpha\beta} + x_3\kappa_{\alpha\beta} - x_3 b^{\lambda}_{(\alpha}\epsilon_{\lambda\beta)} - x_3^2 b^{\lambda}_{(\alpha}\kappa_{\lambda\beta)} + w_{(\alpha;\beta)} \\ &\quad - b_{\alpha\beta}w_3 - x_3 b^{\lambda}_{(\alpha}w_{\lambda;\beta)} + x_3 b^{\lambda}_{\alpha} b_{\lambda\beta} w_3 \\ 2\Gamma_{\alpha 3} &= w_{3,\alpha} + w_\lambda b^{\lambda}_{\alpha} + w_{\alpha,3} - x_3 b^{\beta}_{\alpha} w_{\beta,3} \\ \Gamma_{33} &= w_{3,3} \end{aligned} \tag{16}$$

where the semicolon preceding an index denotes the covariant derivative with respect to the coordinate and parenthesis in the subscript denotes the symmetrization operation (Le, 1999) meaning $a_{(\alpha\beta)} = \frac{1}{2}(a_{\alpha\beta} + a_{\beta\alpha})$. The 2D generalized strain measures $\epsilon_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are defined according to Koiter (1959) and Sanders (1959), such that

$$\begin{aligned} \epsilon_{\alpha\beta} &= \tfrac{1}{2}(A_{\alpha\beta} - a_{\alpha\beta}) \\ \kappa_{\alpha\beta} &= b_{\alpha\beta} - B_{\alpha\beta} + b^{\lambda}_{(\alpha}\epsilon_{\lambda\beta)} \end{aligned} \tag{17}$$

where $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are the first and second fundamental forms of the deformed surface, respectively.

3.2. Asymptotic analysis

For performing the asymptotic analysis, we estimate the following orders of the fundamental variables:

$$\epsilon_{\alpha\beta} \sim h\kappa_{\alpha\beta} \sim \frac{w_i}{h} \sim O(\epsilon), \quad hb_{\alpha\beta} \sim hb^{\alpha}_{\beta} \sim hb^{\alpha\beta} \sim hk_i \sim O(k) \tag{18}$$

The estimation of the orders of the unknown warping functions at each step of asymptotic analysis are more to maintain the logical coherency of the asymptotic analysis rather than necessity, we solve the unknowns and then check whether their order matches with our initial estimates. For strip like open sections there is a relative order difference between two initial curvatures k_2 and k_3, but we have not differentiated them for the simplicity of derivation, rather we will take this into consideration when we assign numerical values for k_2 and k_3 to make sure both parameters are small as required by the theory. For closed section, all three of pre-twist and pre-curvatures are of $O(k)$. Here, $\epsilon \ll 1$ and $k \ll 1$ are small parameters for book keeping purpose. From Eq. (15), we first collect terms which are of order $(\frac{h}{c})^0$, in that collection we seek for terms which are of order $(\frac{c}{l})^0$. In the resulting terms shown in Eq. (16) we keep terms which are of $O(\epsilon)$ and $O(\epsilon k)$. Three-dimensional material tensor E^{ijkl}, correct up to the first order of initial curvatures and twists, contributes terms of $O(\mu)$ and $O(\mu k)$, so the resulting strain energy with first-order correction to initial curvatures and twists contains terms of $O(\mu\epsilon^2)$ and $O(\mu\epsilon^2 k)$.

The zeroth-order strain energy only contains terms up to $O(\mu\epsilon^2)$ contributed by 3D strains of $O(\epsilon)$ given as follows:

$$\Gamma^0_{\alpha\beta} = \epsilon_{\alpha\beta} + x_3\kappa_{\alpha\beta}, \quad 2\Gamma^0_{\alpha 3} = w_{\alpha,3}, \quad \Gamma^0_{33} = w_{3,3} \tag{19}$$

The 3D strain energy of the thin-walled structure can be expressed as

$$\begin{aligned} J &= \frac{1}{2}\int_v \Gamma^T D \Gamma \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \\ &= \frac{1}{2}\int_s \left\langle \Gamma^T D \Gamma (1 - 2x_3 H + x_3^2 K)\sqrt{a} \right\rangle \mathrm{d}x_2 \, \mathrm{d}x_3 \end{aligned} \tag{20}$$

where v is the volume occupied by the 3D body in the undeformed configuration, s is the undeformed surface, $a = \mathrm{Det}(a_{\alpha\beta})$, and

$$\Gamma = \lfloor \Gamma_e \quad 2\Gamma_s \quad \Gamma_t \rfloor^T \tag{21}$$

$$\Gamma_e = \lfloor \Gamma_{11} \quad 2\Gamma_{12} \quad \Gamma_{22} \rfloor^T, \qquad 2\Gamma_s = \lfloor 2\Gamma_{13} \quad 2\Gamma_{23} \rfloor^T, \qquad \Gamma_t = \Gamma_{33} \tag{22}$$

The strain energy per unit area, or the strain energy for the deformation of the normal-line element, can be written as

$$U = \frac{1}{2}\left\langle \Gamma^T D \Gamma (1 - 2x_3 H + x_3^2 K)\sqrt{a} \right\rangle \tag{23}$$

where D is the 6×6 material matrix condensed from the fourth-order material tensor E^{ijkl} defined in the base of $\mathbf{g}_i$. For isotropic material, the components of material tensor can be written explicitly in the base of $\mathbf{g}_i$ as

$$E^{ijkl} = \lambda g^{ij} g^{kl} + \mu (g^{ik} g^{jl} + g^{il} g^{jk}) \tag{24}$$

where λ and μ are Lame constants.

3.2.1. Transformation of the material tensor

For a general anisotropic material, E^{ijkl} are functions of 21 independent material constants and layup angles. Furthermore, E^{ijkl} are also functions of initial curvatures and twist, which means that the material matrix D may be expanded asymptotically as

$$D(1 - 2x_3 H + x_3^2 K)\sqrt{a} = D_0 + D_1 + O(\mu k) \tag{25}$$

where D_0 is of the order of the elastic constants μ, D_1 is of the order μk, and $O(\mu k)$ represents terms of order higher than μk. For laminated composite structures, an orthonormal system (say, with base vectors $\mathbf{c}_i$) should be specified to determine the layup angles. Here we have assumed that ply planes all through the thickness are parallel to the plane formed by $\mathbf{b}_1 \times \mathbf{a}_2$ (justified by the thin-wall assumption) and ply angle is specified with

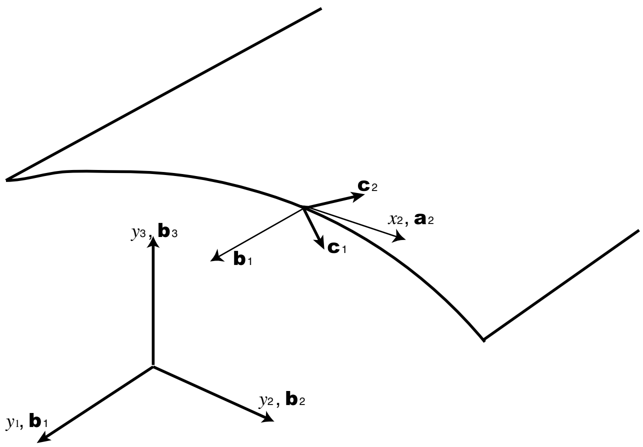

Fig. 2. Schematic of material coordinate system.

respect to the anticlock positive rotation of $\mathbf{c}_1$ from $\mathbf{b}_1$, see the simple sketch in Fig. 2. The explicit expressions of the base vectors, $\mathbf{c}_i$, forming the material coordinate system are given as

$$\begin{aligned}\mathbf{c}_1 &= \mathbf{b}_1\cos\theta + \mathbf{a}_2\sin\theta\\ \mathbf{c}_2 &= -\mathbf{b}_1\sin\theta + \mathbf{a}_2\cos\theta\\ \mathbf{c}_3 &= \mathbf{c}_1\times\mathbf{c}_2\end{aligned}\tag{26}$$

where θ is the specified ply angle. The case of varying ply angles can be treated by dividing the domain of integration through the thickness into constant θ zones. Next, we need to transform the material properties given in the base of $\mathbf{c}_i$ into the base of $\mathbf{g}_i$, which can be achieved using the following transformation rule for fourth-order tensors:

$$\bar{E}^{ijkl} = E^{mnpq}(\mathbf{c}_m\cdot\mathbf{g}^i)(\mathbf{c}_n\cdot\mathbf{g}^j)(\mathbf{c}_p\cdot\mathbf{g}^k)(\mathbf{c}_q\cdot\mathbf{g}^l)\tag{27}$$

where $\bar{E}^{ijkl}$ are the values to be used to construct the material matrix D in Eq. (23), and E^{mnpq} are the components of the material tensor in the base of $\mathbf{c}_i$. It is important to note that Eq. (24) can be derived from Eq. (27) correct up to $O(\mu k)$ by considering isotropic material as orthotropic material having same material properties in three material directions.

According to the VAM, it is sufficient to find the zeroth-order warping for the purpose of obtaining an energy asymptotically correct through the first order of initial curvatures and twist, which is the focus of the present work. The zeroth-order energy per unit area can be written as

$$2U_0 = \left\langle \begin{Bmatrix}\Gamma_e^0\\ 2\Gamma_s^0\\ \Gamma_t^0\end{Bmatrix}^T \begin{bmatrix} D_e & D_{es} & D_{et}\\ D_{es}^T & D_s & D_{st}\\ D_{et}^T & D_{st}^T & D_t\end{bmatrix} \begin{Bmatrix}\Gamma_e^0\\ 2\Gamma_s^0\\ \Gamma_t^0\end{Bmatrix}\right\rangle\tag{28}$$

where D_e, D_{es}, D_{et}, D_s, D_{st}, D_t are the corresponding partition matrices of D_0. Minimizing Eq. (28), one can solve for $2\Gamma_s^0$ and Γ_t^0 as

$$2\Gamma_s^0 = -D_s^{*-T}D_{es}^{*T}\Gamma_e^0,\quad \Gamma_t^0 = -D_t^{-T}D_{et}^{*T}\Gamma_e^0\tag{29}$$

with

$$D_s^* = D_s - D_{st}D_t^{-1}D_{st}^T,\quad D_{es}^* = D_{es} - D_{et}D_t^{-1}D_{st}^T,\quad D_{et}^* = D_{et} - D_{es}^*D_s^{*-1}D_{st}\tag{30}$$

From Eqs. (13), (19), and (29), one can solve for zeroth-order warping field. Substituting the warping functions into Eq. (28), one can obtain the strain energy per unit area asymptotically correct up to $O(\mu\epsilon^2)$ as

$$2U_0 = \begin{Bmatrix}\epsilon\\ \kappa\end{Bmatrix}^T \begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{B}^T & \mathcal{D}\end{bmatrix} \begin{Bmatrix}\epsilon\\ \kappa\end{Bmatrix}\tag{31}$$

where

$$\epsilon = \lfloor \epsilon_{11}\quad 2\epsilon_{12}\quad \epsilon_{22}\rfloor^T,\quad \kappa = \lfloor \kappa_{11}\quad 2\kappa_{12}\quad \kappa_{22}\rfloor^T\tag{32}$$

$$\mathcal{A} = \langle D_\parallel\rangle\quad \mathcal{B} = \langle x_3 D_\parallel\rangle\quad \mathcal{D} = \langle x_3^2 D_\parallel\rangle\tag{33}$$

and

$$D_\parallel = D_e - D_{es}D_s^{*-T}D_{es}^{*T} - D_{et}D_t^{-T}D_{et}^{*T}\tag{34}$$

Substituting the solved warping functions into Eq. (16), one can obtain the 3D strains asymptotically correct up to the first order of initial curvatures and twist, which can be symbolically written as

$$\Gamma = \Gamma_0 + \Gamma_1\tag{35}$$

Using the above equation along with Eqs. (25) and (23), the strain energy per unit area of the first order of the initial twist and curvatures can be calculated as

$$2U_1 = \langle \Gamma^{0T} D_1 \Gamma^0 - 2x_3 H \Gamma^{0T} D_0 \Gamma^0 + 2\Gamma^{1T} D_0 \Gamma^0 \rangle = \begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix}^T \begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{B}_1^T & \mathcal{D}_1 \end{bmatrix} \begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix} \tag{36}$$

Up to this point, we have successfully reduced the original 3D strain energy to a 2D shell strain energy which has an accuracy asymptotically correct through the first order of the initial twist and curvatures.

4. Dimensional reduction from 2D to 1D

The previously obtained model represented in Eqs. (31) and (36) is an asymptotically correct classical shell model with geometric correction through the first order due to initial curvatures and twist of the beam reference line. However, engineering practice often requires a more simplified model to design and analyze thin-walled beams. We need to proceed further to reduce the shell model to a 1D beam model.

4.1. Deformed kinematics

The position vector of the deformed reference surface can be expressed as

$$\mathbf{R}(x_1, x_2) = \mathbf{R}_0 + y_a(x_2)\mathbf{B}_a + v_i(y_1, y_2, y_3)\mathbf{B}_i \tag{37}$$

where $\mathbf{B}_i$ are the unit vectors associated with y_i for the deformed configuration, $\mathbf{R}_0$ is the position vector for a point on the beam reference line. Both $\mathbf{B}_i$ and $\mathbf{R}_0$ are functions of y_1. Warping functions v_i are introduced to account for the difference between $\mathbf{R}$ and those represented by $\mathbf{R}_0$ and $\mathbf{B}_i$. The following four constraints are introduced to ensure a unique mapping between $\mathbf{R}$ and $(\mathbf{R}_0, \mathbf{B}_i)$.

$$\langle\langle v_i \rangle\rangle = 0 \quad \langle\langle y_3 v_2 - y_2 v_3 \rangle\rangle = 0 \tag{38}$$

with the double angle-brackets denoting the definite integral along the contour of beam sections.

From Eq. (37), one can calculate the base vectors of the deformed shell surface $\mathbf{A}_i$ based on their definitions, Eq. (11). Then one can obtain the fundamental forms of this surface and finally derive the shell strain measures in terms of beam quantities and warping functions from Eq. (17).

4.2. Asymptotic analysis

For the purpose of asymptotic analysis we estimate the following orders of the fundamental variables.

$$\gamma_{11} \sim h\kappa_i \sim \frac{v_i}{c} \sim O(\epsilon) \tag{39}$$

where (γ_{11}, κ_i) are beam strain measures and v_i are warpings defined on the wall mid-surface. The estimation of the orders of the unknown warping functions, v_i, are more to maintain the logical coherency of the asymptotic analysis rather than necessity, we solve the unknowns and check whether they match with our initial estimate. By neglecting all nonlinear terms with respect to the beam strain measures and warping functions v_i, one can find the 2D shell strain measures asymptotically correct through the first order of initial curvatures and twist. To avoid lengthy formulas, here, we only present the shell strain measures asymptotically correct to the zeroth-order of initial curvatures and twist. Here one thing to observe, functions $\dot{y}_a \sim O(1)$ and $R_n \sim O(h)$.

$$\begin{aligned}
\epsilon_{11}^0 &= \gamma_{11} + \kappa_2 y_3 - \kappa_3 y_2 \\
2\epsilon_{12}^0 &= \dot{v}_1 + \kappa_1 R_n \\
\epsilon_{22}^0 &= \dot{y}_a \dot{v}_a \\
\kappa_{11}^0 &= \kappa_a \dot{y}_a \\
2\kappa_{12}^0 &= -2\kappa_1 + \frac{b_{22}}{2}(\dot{v}_1 + \kappa_1 R_n) \\
\kappa_{22}^0 &= (\dot{y}_3 \dot{v}_2 - \dot{y}_2 \dot{v}_3)_{,2}
\end{aligned} \tag{40}$$

where γ_{11} is the extensional strain and κ_1 the torsional strain and κ_a the bending strain in the y_a direction.

To obtain the strain energy defined along the beam axis through the first order of initial twist and curvatures, one needs to solve for v_i of the zeroth-order approximation. Substituting the 2D strains in Eq. (40) into the zeroth-order shell energy, Eq. (31), one can solve for the warping functions subject to the constraints in Eq. (38). For the convenience of calculation, one can express the zeroth-order 2D shell strains in matrix form as

$$\begin{Bmatrix} \epsilon \\ \kappa \end{Bmatrix} = P\varepsilon + T\psi \tag{41}$$

with

$$P = \begin{bmatrix} 1 & 0 & y_3 & -y_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \dot{y}_2 & \dot{y}_3 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \frac{b_{22}}{2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{42}$$

$$\epsilon = [\epsilon_{11}^0 \quad 2\epsilon_{12}^0 \quad \epsilon_{22}^0]^T, \quad \kappa = [\kappa_{11}^0 \quad 2\kappa_{12}^0 \quad \kappa_{22}^0]^T \tag{43}$$

$$\varepsilon = [\gamma_{11} \quad \kappa_1 \quad \kappa_2 \quad \kappa_3]^T, \quad \psi = [2\epsilon_{12}^0 \quad \epsilon_{22}^0 \quad \kappa_{22}^0]^T \tag{44}$$

The only unknown degrees of freedom exist in ψ and the constraints in Eq. (38) are only needed to recover v_i and have no effect on the arbitrariness of ψ. Denoting the stiffness obtained in Eq. (31) as K, one can express the zeroth-order energy per unit length of the beam axis using Eq. (41) as

$$2\Pi_0 = \langle\langle (P\varepsilon + T\psi)^T K (P\varepsilon + T\psi)\rangle\rangle \tag{45}$$

For open sections, there is no additional constraints on ψ. The minimization problem can be carried out in straightforward manner, yielding

$$\psi = -(T^T K T)^{-1} T^T K P \varepsilon \tag{46}$$

Substituting Eq. (46) back in Eq. (45), we can get the expression of the zeroth-order strain energy in terms of beam strains and given as

$$2\Pi_0 = \varepsilon^T \langle\langle P^T [K - KT(T^T K T)^{-1} T^T K] P\rangle\rangle \varepsilon \tag{47}$$

The zeroth-order warpings can be solved from the expression of ψ. For closed sections, four additional constraints should be applied to ensure the uniqueness of the displacement field. Following Volovoi and Hodges (2000), we have

$$\langle\langle v_{i,2}\rangle\rangle = 0, \quad \langle\langle \kappa_{22}^0\rangle\rangle = 0 \tag{48}$$

The constraints can be transformed into matrix form as

$$\langle\langle \phi\psi - L\varepsilon\rangle\rangle = 0 \tag{49}$$

with

$$\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dot{y}_2 & -y_3 \\ 0 & \dot{y}_3 & y_2 \\ 0 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & R_n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{50}$$

Introducing Lagrange multipliers, the functional to be minimized has the form

$$2\Lambda = \langle\langle (P\epsilon + T\psi) K (P\epsilon + T\psi) + 2\lambda^T \phi\psi\rangle\rangle \tag{51}$$

From which one can solve for ψ

$$\psi = -(T^T K T)^{-1} (T^T K P \epsilon + \phi^T \lambda) \tag{52}$$

Substituting the above equation into the constraints in Eq. (49), one can solve for λ as

$$\lambda = -\langle\langle \phi (T^T K T)^{-1} \phi^T \rangle\rangle^{-1} \langle\langle \phi (T^T K T)^{-1} T^T K P + L \rangle\rangle \varepsilon \tag{53}$$

Using Eq. (53) one can get the expression of ψ from Eq. (52) and subsequently the zeroth-order warping functions. Finally one can obtain the strain energy per unit length asymptotically correct up to the first order of initial twist and curvatures.

5. Applications

To validate and demonstrate the applications of the present model, we will study both initially twisted and curved beams with open sections, such as strips, and closed sections, such as box-beams. The results will be compared with available results in the literature and VABS, a finite element based, general-purpose, cross-sectional analysis code (Yu et al., 2002). First, it is verified that if we set $k_2 = k_3 = 0$, all the formulas and results in Yu et al. (2005) are reproduced.

5.1. Isotropic cases

5.1.1. Open section

The first example is an isotropic strip with width c, thickness h, initial twist k_1, and initial curvatures k_2 and k_3. The material has E as its Young's modulus and v as its Poisson's ratio. Using the formulas listed in previous section, we can obtain the stiffness values up to the first order of initial curvatures and twist for the classical beam model for an isotropic strip as:

$$s_{11} = Ech, \qquad s_{22} = \frac{Ech^3}{6(1+v)}, \qquad s_{33} = \tfrac{1}{12}Ech^3, \qquad s_{44} = \tfrac{1}{12}Ec^3h$$
$$s_{12} = \tfrac{1}{12}Ec^3h[1 - 3\left(\frac{h}{c}\right)^2]k_1, \qquad s_{13} = -\tfrac{1}{12}Ech^3(1+v)k_2, \qquad s_{14} = -\tfrac{1}{12}Ec^3h(1+v)k_3$$

where s_{11} is the extensional stiffness, s_{22} torsional stiffness, s_{33} bending stiffness in the x_2 direction, s_{44} bending stiffness in the x_3 direction, s_{12} the extension–twist coupling term, and s_{13}, s_{14} the extension–bending coupling terms. The relations between the couplings and corresponding diagonal stiffness terms are the same as those for initially twisted and curved isotropic solid beams obtained in Berdichevsky and Starosel'skii (1985) and Berdichevsky and Starosel'skii (1983) without taking advantage of the smallness of wall thickness, which are listed here for completeness.

$$\begin{aligned} s_{12} &= [s_{33} + s_{44} - 2(v+1)s_{22}]k_1 \\ s_{13} &= -(1+v)s_{33}k_2 \\ s_{14} &= -(1+v)s_{44}k_3 \end{aligned} \tag{54}$$

These relations are also verified in other works, including Cesnik et al. (1996) and Hodges (1999). The fact that our model can reproduce such relations clearly demonstrates the validity of the model for isotropic strips in particular, open sections in general.

5.1.2. Closed section

Usually, common approaches to thin-walled beam theories uses different models for open sections and closed-sections. However, our model provides a unified treatment for all thin-walled beams including both closed and open sections. To validate the present model for isotropic closed sections, we study an isotropic box-beam with width a, height b, wall thickness h, initial twist k_1, and initial curvatures k_2 and k_3. The cross-sectional stiffness constants for this beam are:

$$
\begin{aligned}
s_{11} &= 2(a+b)Eh \\
s_{22} &= \frac{Eh(2abh^2 + b^2h^2 + 3a^2b^2 + a^2h^2)}{3(a+b)(1+v)} \\
s_{33} &= \frac{Eh(b^3 + 3ab^2 + ah^2)}{6} \\
s_{44} &= \frac{Eh(a^3 + 3a^2b + bh^2)}{6} \\
s_{12} &= \frac{Ehk_1(a^4 + 4ba^3 - 3a^2(2b^2 + h^2) + a(4b^3 - 6bh^2) + b^4 - 3b^2h^2)}{6(a+b)} \\
s_{13} &= -\frac{Ehk_2(b^3 + 3ab^2 + ah^2)(1+v)}{6} \\
s_{14} &= -\frac{Ehk_3(a^3 + 3a^2b + bh^2)(1+v)}{6}
\end{aligned}
$$

The diagonal stiffness constants are the same as the prismatic beams. The elastic couplings s_{12}, s_{13}, and s_{14} due to existence of initial twist and curvatures again agree with the relations in Eq. (54).

5.2. Composite cases

To show the validity of the present model to predict the cross-sectional properties of thin-walled structures made of composite material, we will study a composite strip and a composite box-beam with different initial curvatures and twist. The composite material has the following mechanical properties:

$$
\begin{aligned}
&E_{11} = 25 \times 10^6\text{psi}, \quad E_{22} = E_{33} = 10 \times 10^6\text{psi}, \quad G_{12} = G_{12} = 5 \times 10^6\text{psi} \\
&G_{23} = 2 \times 10^6\text{psi}, \quad v_{12} = v_{13} = v_{23} = 0.25
\end{aligned}
$$

5.2.1. Open section

First, we consider an example of a single layer composite strip of width $c = 2$ in, $h = 0.2$ in and layup orientation 15°. The straight strip exhibits twist-bending coupling (s_{23}) due to anisotropy of composite materials. When the strip has initial twist and curvatures, other couplings due to this geometry change in addition to the material coupling might also be generated. For example, when $k_1 = 0.1$/in, as shown in Table 1 (only non-zero values are listed), a significant extension–twist coupling (s_{12}) will exist in the beam constitutive model. Although s_{13} is also appearing in the model, its effect in the global behavior of strip is negligible. There are not many results in the literature for initially twisted and curved composite, thin-walled beams we can use to validate the present model. A perfect validation method is VABS, a finite-element based, 2D, cross-sectional analysis. VABS performs the dimensional reduction directly from 3D to 1D without thin-walled assumption and a 2D finite element mesh should be used as input for VABS to carry out the numerical analysis. However, the present model analytically splits dimensional reduction into two 1D analyses by taking advantage of the thinness of walls. The present model has a fair agreement with VABS as shown in Tables 1–4. The % error is calculated as $\frac{|\text{PRESENT}|-|\text{VABS}|}{|\text{VABS}|} \times 100$. As explained in Yu et al. (2005), the big differences in the torsion related

Table 1
Composite strip with $k_1 = 0.1$/ in and $k_2, k_3 = 0$

	PRESENT	VABS	% Error
s_{11}	8.59859×10^6	8.59859×10^6	0.00
s_{12}	2.80435×10^5	2.77920×10^5	0.90
s_{13}	-6.13958×10^2	2.57001×10^1	–
s_{22}	2.99683×10^4	2.78754×10^4	7.51
s_{23}	-7.62111×10^3	-7.08888×10^3	7.51
s_{33}	3.06001×10^4	3.04647×10^4	0.44
s_{44}	2.8662×10^6	2.8662×10^6	0.00

Table 2
Composite strip with k_1, $k_3 = 0$ and $k_2 = 0.1$/in

	PRESENT	VABS	% Error
s_{11}	8.59859×10^6	8.59908×10^6	−0.01
s_{12}	1.70451×10^3	1.58555×10^3	7.50
s_{13}	-3.97771×10^3	-3.94773×10^3	0.76
s_{22}	2.99683×10^4	2.78753×10^4	7.51
s_{23}	-7.62111×10^3	-7.08954×10^3	7.50
s_{33}	3.06001×10^4	3.04672×10^4	0.44
s_{44}	2.8662×10^6	2.86637×10^6	−0.01

Table 3
Composite strip with k_1, $k_2 = 0$ and $k_3 = 0.01$/in

	PRESENT	VABS	% Error
s_{11}	8.59859×10^6	8.5988798×10^6	−0.01
s_{14}	-3.63636×10^4	-3.63663×10^4	0.00
s_{22}	2.99683×10^4	2.78801×10^4	7.49
s_{23}	-7.62111×10^3	-7.08978×10^3	7.49
s_{33}	3.06001×10^4	3.04665×10^4	−0.44
s_{44}	2.8662×10^6	2.88669×10^6	−0.71

Table 4
Composite strip with $k_1 = 0.1$/in, $k_2 = 0.1$/in and $k_3 = 0.01$/in

	PRESENT	VABS	% Error
s_{11}	8.59859×10^6	8.59951×10^6	−0.01
s_{12}	2.82140×10^5	2.79582×10^5	0.91
s_{13}	-4.59166×10^3	-3.9205×10^3	17.12
s_{14}	-3.63636×10^4	-3.63581×10^4	0.02
s_{22}	2.99683×10^4	2.7949×10^4	7.22
s_{23}	-7.62111×10^3	-7.27201×10^3	4.80
s_{24}	0	-3.25686×10^3	–
s_{33}	3.06001×10^4	3.04612×10^4	0.46
s_{34}	0	-9.9317×10^1	–
s_{44}	2.8662×10^6	2.86653×10^6	−0.01

terms are due to the thin-walled assumption. When h/c becomes smaller, the present model agrees with VABS better. When the strip is initially curved in the soft direction (see Table 2), both extension–twist (s_{12}) and extension–bending (s_{13}) coupling will be generated. However, when the strip is initially curved in the stiff direction (see Table 3), only extension–bending coupling due to this geometry change. It is noted that we purposely assigned a smaller number to k_3 because it is physically more difficult to bend the strip along the stiff direction, which means bending the strip the same amount along the stiff direction will have a much bigger effect on the global behavior than bending the strip in the soft direction. When the strip is initially twisted and curved in both directions, all the fundamental deformation modes of the strip are coupled with each other as shown by VABS results in Table 4. There are two couplings (s_{24} and s_{34}) cannot be captured by the present model, which are found out to be proportional to k_3. The main reason is that some higher order terms of k_i are captured by VABS because it does not expand $\sqrt{g}$ (g is the determinant of 3D metric tensor) asymptotically for convenience of numerical implementation. As limited by the theory itself, the present model is incapable of capturing effect beyond $O(\mu\epsilon^2 k)$ in the energy. It is also suggested one should be cautions if the present model is used to analyze strips having a not so small curvature along the stiff direction.

5.2.2. Closed section

A box-beam with width $a = 0.923$ in, depth $b = 0.5$ in and thickness $h = 0.03$ in is used to demonstrate the application of the present theory to closed section. The box-beam is made with material properties and ply orientation same as the composite strip example for each wall composing the box. The results for a composite box-beam have much better agreements with VABS solution, as shown in Tables 5–8. The main reason is that for closed section all the pre-twist and pre-curvature are of the same order. Of course, there are still some minor couplings (s_{23}, s_{24}, and s_{34}) are not well captured by the present model when the box-beam become twisted and curved in both directions. The higher-order terms of k_i incorporated in VABS are magnified when all of these three geometry parameters k_i exist.

Table 5
Composite box-beam with $k_1 = 0.1$/in and $k_2, k_3 = 0$

	PRESENT	VABS	% Error
s_{11}	1.95948×10^6	1.95892×10^6	0.03
s_{12}	-6.67131×10^4	-6.70212×10^4	−0.46
s_{22}	4.96042×10^4	5.05872×10^4	−1.94
s_{33}	8.54112×10^4	8.56976×10^4	−0.33
s_{44}	2.13321×10^5	2.14066×10^5	−0.35

Table 6
Composite box-beam with $k_1, k_3 = 0$ and $k_2 = 0.1$/in

	PRESENT	VABS	% Error
s_{11}	1.95948×10^6	1.96094×10^6	−0.07
s_{12}	-7.91357×10^4	-7.92438×10^4	−0.14
s_{13}	-1.26627×10^4	-1.24577×10^4	1.65
s_{22}	5.06059×10^4	5.15851×10^4	−1.90
s_{23}	9.62765×10^2	9.32068×10^2	3.29
s_{33}	8.79354×10^4	8.82562×10^4	−0.36
s_{44}	2.21919×10^5	2.22882×10^5	−0.43

Table 7
Composite box-beam with k_1, $k_2 = 0$ and $k_3 = 0.1$/in

	PRESENT	VABS	% Error
s_{11}	1.95948×10^6	1.96329×10^6	0.03
s_{12}	-7.91357×10^4	-7.94133×10^4	−0.46
s_{14}	-3.19605×10^4	-3.19663×10^4	−0.46
s_{22}	5.06059×10^4	5.16996×10^4	−1.94
s_{24}	2.42887×10^3	2.41876×10^3	−1.94
s_{33}	8.79354×10^4	8.83854×10^4	−0.33
s_{44}	2.21919×10^5	2.23348×10^5	−0.35

Table 8
Composite box-beam with k_1, k_2, $k_3 = 0.1$/in

	PRESENT	VABS	% Error
s_{11}	1.95948×10^6	1.9642×10^6	−0.24
s_{12}	-6.67131×10^4	-6.72213×10^4	−0.76
s_{13}	-1.26627×10^4	-1.23988×10^4	2.13
s_{14}	-3.19605×10^4	-3.14324×10^4	1.68
s_{22}	4.96042×10^4	5.07168×10^4	−2.19
s_{23}	9.62765×10^2	8.29659×10^2	16.04
s_{24}	2.42887×10^3	1.95709×10^3	24.10
s_{33}	8.54112×10^4	8.59273×10^4	−0.60
s_{34}	0.0000	2.77503×10^2	–
s_{44}	2.13321×10^5	2.14774×10^5	−0.68

6. Conclusion

An asymptotically correct model, capable of analyzing extension, torsion, and bending in two directions, has been developed for composite thin-walled beams with initial twist and curvatures. The final beam model is systematically reduced from the original 3D elasticity model using the variational asymptotic method without invoking any *ad hoc* kinematical assumptions. Closed-form solutions have been obtained for isotropic strips and box-beams to demonstrate that the developed model is capable of providing 1D beam models asymptotically correct up to the first order of initial curvatures and twist for isotropic thin-walled beams with open or closed sections. For general thin-walled beams made of composite materials, closed-form expressions become too lengthy to be presentable. Nevertheless, the numerical values coming out of a symbolic manipulator (we used Mathematica), have very good agreement with those obtained from VABS, a general-purpose cross-sectional analysis tool. Some of the mismatches arising in the minor coupling terms are due to the thin-walled assumptions and higher order terms of k_i incorporated in VABS because VABS does not expand $\sqrt{g}$ asymptotically with respect to curvatures k_2 and k_3, as a result there are some higher order effect adding to the slight disagreement.

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